JOURNAL OF COMBINATORIAL THEORY, Series B 26, 150–153 (1979)

A Threshold Property for Intersections in a Finite Set

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Communicated by the Editors

Received June 4, 1976

Let F be any family of subsets of a finite set E and let n be an integer, $n < |F|$. Under what condition does the knowledge of cardinals of m-intersections in F, for all $m \leqslant n$, univocally determine the cardinal of any intersection in F, and what is the minimal condition? We give a complete answer to that. For any n, this determination property is satisfied by n if and only if $|E| < 2^n$, without further condition on F.

1. DEFINITIONS

Let E be a finite set, let F be a family of subsets of E, and let m be an integer. We call an *m-intersection in F* any intersection of m sets which belong to F.

Let $F = \{A_1, ..., A_p\}$ and $F' = \{A'_1, ..., A'_p\}$ be two families of subsets of E, and let n be an integer such that $n \leqslant p$. We say that F and F' satisfy the *intersection condition of rank n* (abbreviated i.c. of rank n) iff for every $m \leqslant n$ and every m-set $\{i_1, ..., i_m\}$ included in $\{1, ..., p\}$, we have $|A_{i_1} \cap \cdots \cap A_{i_m}| = |A'_{i_1} \cap \cdots \cap A'_{i_m}|$.

2. RESULTS

THEOREM. *Let E be a set and let n be an integer such that $2^{n-1} \leqslant |E| < 2^n$; then, for any $p \geqslant n$ and any two families $F = \{A_1, ..., A_p\}$ and $F' = \{A'_1, ..., A'_p\}$ of subsets of E, if F and F' satisfy the i.c. of rank n, then they satisfy the i.c. of rank p too; but the i.c. of rank $(n - 1)$ is not sufficient to imply the i.c. of rank p.*

LEMMA 1. *Let E be a set and let n be a fixed integer; if $|E| < 2^n$, then for any family F of subsets of E, the knowledge of cardinals of m-intersections in F, for every $m \leqslant n$, univocally determines the cardinals of p-intersections for any p ($p \leqslant |F|$).*

0095-8956/79/020150-04$02.00/0

Proof. By induction over n. It is trivial for $n = 1$ and it is easy to see for $n = 2$. Assume the result for $(n - 1)$, and let $F = \{A_1, ..., A_{n+1}\}$ be a family of subsets of a set E such that $|E| < 2^n$; we suppose that the cardinals of the m-intersections are known for every $m \leqslant n$, and we prove that the cardinal of $A_1 \cap \cdots \cap A_{n+1}$ is univocally determined. We separate this into two cases.

Case 1. There is a subset A_i such that $|A_i| < 2^{n-1}$, let A_{i_0} be this A_i. Then, we consider the family $A_i \cap A_{i_0}$, $i = 1, ..., n + 1$, $i \neq i_0$, and we put $A'_i = A_i \cap A_{i_0}$. Let $p \leqslant n - 1$, each p-intersection in this new family is in fact a $(p + 1)$-intersection in F, and hence its cardinal is known; by the induction hypothesis, the cardinal of the intersection of the (A'_i)'s is univocally determined, and this intersection is identical with $A_1 \cap A_2 \cap \cdots \cap A_{n+1}$.

Case 2. For every i $(= 1, ..., n + 1)$, we have $|A_i| \geqslant 2^{n-1}$, and hence $|E \backslash A_i| < 2^{n-1}$. We consider, for each $i = 2, ..., n + 1$, the set $A'_i = A_i \cap (E \backslash A_1)$ and let F' be the family of the (A'_i)'s. If C' is a p-intersection in F', for $p \leqslant n - 1$, we can write $C' = C \cap (E \backslash A_1)$, where C is a p-intersection in F. We have $|C'| = |C| - |C \cap A_1|$, and since $|C|$ and $|C \cap A_1|$ are known by hypothesis, we can determine $|C'|$. Using the induction hypothesis for F', we claim that $|A_2 \cap \cdots \cap A_{n+1} \cap (E \backslash A_1)|$ is determined. Let's note as c this cardinal; to finish, we can see that $|A_1 \cap A_2 \cap \cdots \cap A_{n+1}| = |A_2 \cap \cdots \cap A_{n+1}| - c$.

LEMMA 2. *Let an integer n and let E be a set, with* Card $(E) = k$. *If $k = 2^{n-1}$, then there exist two families of subsets of E, $F = \{A_1, ..., A_n\}$ and $F' = \{A'_1, ..., A'_n\}$, such that F and F' satisfy the i.c. of rank $n - 1$, while $|A_1 \cap \cdots \cap A_n| \neq |A'_1 \cap \cdots \cap A'_n|$ (more exactly, $|A_1 \cap \cdots \cap A_n| = 1$ but $A'_1 \cap \cdots \cap A'_n = \varnothing$).*

Proof. We use the following relations between the binomial coefficients:

$$\sum_{\text{even } p} \binom{m}{p} = \sum_{\text{odd } p} \binom{m}{p} = 2^{m-1}.$$

Now, we are going to build two families of subsets of a set E, with the required properties.

We start, on one hand, from a family $\{A_i{}^0, i = 1, ..., n\}$ such that all the $(A_i{}^0)$'s are identical with the same singleton, and on the other hand, from a family $\{A_i'^0 / i = 1, ..., n\}$ such that all the $(A_i'^0)$'s are empty. We increase alternately the $(A_i{}^0)$'s and the $(A_i'^0)$'s, in such a manner that first the i.c. of rank $(n - 1)$ holds, and then for the rank $(n - 2), ...,$ until the rank 1.

We inductively describe this construction.

Step 1. The $(n - 1)$-intersections in the family of $A_i{}^0$ have one element each (the same). We increase each $A_i'^0$ so as to form $A_i'^1$; for this, we consider each $(n-1)$-set $\bar{a} = \{i_1, ..., i_{n-1}\}$ included in $\{1, ..., n\}$, and we add a same element $x(\bar{a})$ to $A'^0_{i_1}, ...,$ and to $A'^0_{i_{n-1}}$, but with $x(\bar{a}) \neq x(\bar{b})$ for $\bar{a} \neq \bar{b}$. The number of elements so added to $\bigcup_{i=1}^{n} A'^0_i$ $(= \varnothing)$, is equal to $\binom{n}{n-1} = \binom{n}{1} = n$. Each $(n-2)$-intersection in the family of A'^1_i has two elements.

Step 2. Each $(n-2)$-intersection in the family of A_i^0 has only one element. Then we increase A_i^0 so as to form A_i^1, in a similar way as above, but by considering the $(n-2)$-sets included in $\{1, ..., n\}$. The complete number of added elements is here equal to $\binom{n}{2}$. Each $(n-3)$-intersection in the family of A_i^1 has $1 + \binom{3}{1} = 4$ elements, while in the family of A'^1_i, it has three elements. Thus we continue.

Step $2j$. The $(n-2j)$-intersections in the family of A'^j_i have $\binom{2j}{1} + \binom{2j}{3} + \cdots + \binom{2j}{2j-1}$ elements each, while in the family of A^{j-1}_i, they have $1 + \binom{2j}{2} + \binom{2j}{4} + \cdots + \binom{2j}{2j-2}$ elements; hence, we must add one element to each intersection of the latter kind. So, we form the family of A_i^j; the complete number of added elements on this side since the start, is equal to $\binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{2j}$. Each $(n-2j-1)$-intersection in the family of A_i^j has $1 + \binom{2j+1}{2} + \cdots + \binom{2j+1}{2j}$ elements.

Step $2j+1$. In the family of A'^j_i, each $(n-2j-1)$-intersection has only $\binom{2j+1}{1} + \binom{2j+1}{3} + \cdots + \binom{2j+1}{2j-1}$ elements; we must add one element to each intersection of this kind, to form the (A'^{j+1}_i)'s.

This continues, and the end of this construction splits into two cases.

Case 1. n is odd, let $n = 2p + 1$. The construction comes to an end at step $2p$. Each A'^p_i has $\binom{2p}{1} + \binom{2p}{3} + \cdots + \binom{2p}{2p-1}$ elements, while each A^{p-1}_i has $1 + \binom{2p}{2} + \binom{2p}{4} + \cdots + \binom{2p}{2p-2}$ elements. We add one element to each A^{p-1}_i, to form A_i^p. The cardinal of the union of A_i^p, $i = 1, ..., n$, is equal to $1 + \binom{2p+1}{2} + \cdots + \binom{2p+1}{2p} = 2^{2p} = 2^{n-1}$, and it is larger than that of the union of (A'^p_i)'s.

Case 2. n is even, let $n = 2p$. Here the construction come to an end at the step $2p - 1$. The maximal cardinal is that of the union of A'^p_i, $i = 1, ..., n$, and it is equal to $\binom{2p}{1} + \binom{2p}{3} + \cdots \binom{2p}{2p-1} = 2^{2p-1} = 2^{n-1}$.

This completes the proof of Lemma 2.

We can deduce the following proposition.

COROLLARY. *Let E be a finite set and let n be an integer such that $2^{n-1} \leqslant |E| < 2^n$. For any $p \geqslant n$ and any two families $F = \{A_1, ..., A_p\}$ and $F' = \{A'_1, ..., A'_p\}$ of subsets of E, if these families satisfy the i.c. of rank n, then there exists a permutation of E which maps A_i onto A'_i for each $i = 1, ..., p$; otherwise, there is no such permutation.*

Proof. The i.c. of rank n for F and F' is clearly necessary to the existence of such a permutation of E. Now, suppose that F and F' satisfy the i.c. of rank n; then, by the theorem, they satisfy the i.c. of rank p. We consider the boolean algebras B and B' generated by F and F', and we consider the minimal terms (so called complete products) in B and B'. Any minimal term in B is defined by an intersection $(\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in J}(E\backslash A_j))$, where $I \cup J = \{1,..., p\}$ and $I \cap J = \varnothing$, and we say that the previous minimal term in B and $(\bigcap_{i\in I} A'_i) \cap (\bigcap_{j\in J}(E\backslash A'_j))$, the minimal term in B', are homologous. In order to prove that there exists a permutation of E which maps A_i onto A'_i for each $i = 1,..., p$, it is sufficient to show that any two homologous minimal terms have the same cardinality. For this, we can use the following result: $|(\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in P\backslash I}(E\backslash A_j))| = \sum_{K\supset I}(-1)^{|K|-|I|} \mid \bigcap_{k\in K} A_k \mid$, with $P = \{1,..., p\}$ and I any subset of P, according to [1] (in the proof of "sieve formula" from Sylvester's formula, p. 77) or according to a theorem in [2, p. 19].

We use this result to build (in a model theoretic article to appear) torsion groups which have some threshold propriety for the elementary automorphisms (partial maps preserving the truth value of elementary formulas). But, given the purely combinatorial nature of this result, we can hope it may be useful for other questions.

References

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